POLYNOMIAL GAMES

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0. Introduction.

A basis is laid in this paper for a theory of two-person zero-sum games in which the payoff is a polynomial function $P(x,y)$ of the two strategy variables x and y, the latter taking their values from closed, one-dimensional intervals. A somewhat more general category of "polynomial-like" games is examined first: games whose payoff has the form

$$K(x,y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, r_i(x) s_j(y) ,$$

r_i and s_j being any continuous functions. A general discussion of games with continua of strategies appears elsewhere in this volume [2].

Polynomial games are important as a bridge, leading from the discrete games, whose theory has been well explored, to more general classes of infinite games which admit polynomial approximations to their payoff functions. No nontrivial properties of such approximations have been obtained. One immediate observation is that the error in the game value does not exceed the least upper bound of the error in the payoff. A similar uniformity in the approach to the optimal strategies can not be guaranteed in general. The approximative properties of polynomial-like games are presumably superior in some respects to those of polynomial games; but, as will be seen, the results achieved here for the wider class are less sharp.

[1] Portions of this paper were presented to the American Mathematical Society at Columbus, Ohio in December 1948, and at Palo Alto, California, in April 1949.

The feature of polynomial and polynomial-like games which links them to the discrete case is the finite dimensionality of the spaces of mixed strategies. We may study the solutions of a discrete, $m \times n$ game by means of a geometric model involving an $m-1$-dimensional simplex and an n-dimensional convex polyhedral cone (the positive orthant of Euclidean n-space). The corresponding figures for a polynomial game, of degrees m and n in x and y respectively, are the m-dimensional moment space and the $n+1$-dimensional cone of n^{th} degree polynomials non-negative over the interval $0 \leq y \leq 1$.

An extended study of the geometrical properties of the moment spaces and of the non-negative polynomial spaces is to be published elsewhere [6]; a review of many of the results has already appeared. [5] In the present paper the relevant portions are cited (with informal proofs)(§4) and applied (§5) to derive a number of inequalities relating the sets of optimal mixed strategies of the two players in a polynomial game. The results (Theorems 6 and 7) are not as sharp as corresponding results in the discrete case [1], [4] because the moment spaces and the polynomial cones are not polyhedral figures. It seems likely that some new indices characterising these spaces, beyond those considered in this paper, will have to be introduced before the inequalities can be substantially improved.

A possible computational procedure for polynomial games is outlined in the final section (§6).

1. Conical reciprocation in game theory.

A finite dimensional, bilinear game may be described as follows: Player I chooses a point $r = (r_1, \ldots, r_m)$ from a set R lying in Euclidean m-space. Player II chooses a point $s = (s_1, \ldots, s_n)$ from a set S in Euclidean n-space. R and S are bounded, closed, convex. The kernel $A(r,s)$, or payoff from II to I, is given by the matrix (a_{ij}) in the form

$$A(r,s) = \sum_{i,j=1}^{m,n} a_{ij} r_i s_j . \tag{1.1}$$

The minimax theorem for bilinear functions over convex sets[7] asserts that

$$\min_{s \in S} \max_{r \in R} A(r,s) = \max_{r \in R} \min_{s \in S} A(r,s) = v , \tag{1.2}$$

thereby defining the <u>value</u> v of the game. <u>Optimal strategies</u> for the two players are defined to be points r^0, s^0 such that

$$\min_{s \in S} A(r^0,s) = v , \quad \max_{r \in R} A(r,s^0) = v . \tag{1.3}$$

The sets of optimal strategies, as functions of the coefficients (a_{ij}), vary in a semi-continuous manner, described in the following theorem.

THEOREM 1. *Let G be an open set containing the set $R^0(A)$ of optimal strategies of the first player in the game A. Then $\varepsilon = \varepsilon(G) > 0$ exists such that $R^0(B)$ is contained in G for every game B with coefficients (b_{ij}) satisfying*

$$|b_{ij} - a_{ij}| \leq \varepsilon \qquad \text{all } i, \ j.$$

(Compare lemma 6 in [1] .)

Proof; Suppose the contrary. Then a sequence $\{B^{(k)}\}$ of games can be found with

$$\left| b_{ij}^{(k)} - a_{ij} \right| \leq \varepsilon_k \qquad \{\varepsilon_k\} \longrightarrow 0$$

each of which has an optimal strategy $r^{(k)}$ outside of G. The value $v^{(k)}$ of $B^{(k)}$ converges to a value of A. The $r^{(k)}$ have a limit point in the compact region $R - G$. Since each $r^{(k)}$ guarantees the amount $v^{(k)}$ in the game $B^{(k)}$ this limit point is seen to be an optimal strategy in the original game A, and hence lies in $R^0(A)$ G. This contradiction proves the theorem .

We now describe a principle of conical reciprocation which gives us a geometrical approach to the study of the structure of the solutions. This method was used previously in the analysis of discrete and convex kernels [1],[2],[4]. Imbed the set S in $n+1$-space by affixing the coordinate $s_0 = 1$ to each point. Construct from it the closed, convex cone P_S of points λs, $s \in S$, $\lambda \geq 0$. The reciprocal, or conjugate cone, P_S^*, is defined to be the set of points $h = (h_0, \ldots, h_n)$ for which

$$\sum_{j=0}^{n} h_j s_j \geq 0 \qquad \text{all } s \text{ in } P_S$$

P_S^* is a convex, closed cone, and the fundamental duality theorem[9] states that $(P_S^*)^* = P_S$. P_R and P_R^* will denote the analogous cones in $m+1$-space generated by the set R.

The way to visualize the connection between P_S and P_S^* is to consider the latter as a region in the space of all oriented hyperplanes through the origin of $n+1$-space. These hyperplanes may be represented by the homogeneous linear functions

$$H(s) = \sum_{j=0}^{n} h_j s_j = 0, \quad \text{some } h_j \neq 0$$

positive multiples of H being identified. Using the orientation, we may regard them as half-spaces. Then P_S^* is essentially the set of closed half-spaces that contain P_S. The interior of P_S^* is the set of open half-spaces containing P_S (except for the origin). The boundary points of P_S^* are just the supporting hyperplanes to P_S. The points $(h_0, \ldots, h_n)$ themselves lie on the directed normals to the hyperplanes they determine. This description may of course be dualized to give P_S in terms of P_S^*.

We now define:

$$(1.4)\qquad \begin{aligned} a_{00} &= -v , \\ a_{01} = \cdots = a_{0n} &= 0 , \\ a_{10} = \cdots = a_{m0} &= 0 . \end{aligned}$$

The effect of thus augmenting the matrix (a_{ij}) is merely to change the value of the game from v to 0, and to adapt the form (1.1) to the higher dimensional spaces in which the cones are constructed.

For convenience we introduce the operator A and the inner product notation (h, s) for points in $n+1$-space, giving us

$$(Ar, s) \quad \text{for} \quad \sum_{j=0}^{n} \left(\sum_{i=0}^{m} a_{ij} r_i \right) s_j \ .$$

Let AR stand for the image of R under A, plotted in $n+1$-space. Since AR lies within the hyperplane $h_0 = -v$ its dimension is not more than n. (See however §4.)

The dimension of a convex set of Euclidean n-space is the dimension of the smallest linear manifold containing the set. An interior point is one having a (full n-dimensional) neighborhood entirely within the set; the other points are the boundary points. By an inner point of a p-dimensional convex set we shall mean one that has a neighborhood whose intersection with the set is open in the p-dimensional manifold in which the convex set is contained: an inner point is interior if and only if $p = n$. An inner point can be represented as a convex combination of points in the set in such a way that any preassigned point of the set occurs with positive weight.

A hyperplane separates two convex sets if the two closed half-spaces determined by the hyperplane each contain one of the sets. Two convex sets can always be separated if they have no inner point in common (but this is not a necessary condition for separation).

LEMMA 1. <u>*The convex bodies*</u> AR <u>*and*</u> P_S^* <u>*intersect; in fact*</u>

$$AR \cap P_S^* = AR^0 ,$$

<u>*if*</u> R^0 <u>*denotes the set of optimal strategies of player I.*</u>

Proof: By (1.3) and (1.4) any r^0 in R^0 satisfies

$$(Ar^0, s) \geq 0 \quad \text{all } s \text{ in } S .$$

It follows that Ar^0 is a point of P_S^*. Hence

$$AR^0 \subset AR \cap P_S^* .$$

Conversely, for any r in R, if Ar is in P_S^* then

$$(Ar, s) \geq 0 \quad \text{all } s \text{ is } S,$$

and hence such an r is optimal for player I. Thus

$$AR \cap P_S^* \subset AR^0 .$$

The proof is completed with the observation that the existence of optimal strategies assures that AR^0 is not empty.

LEMMA 2. <u>*The convex bodies*</u> AR <u>*and*</u> P_S^* <u>*can be separated by a hyperplane; in fact, the separating hyperplanes correspond one-one with the optimal strategies of player II.*</u>

Proof: Let S^0 denote the set of optimal strategies of player II. Every s^0 in S^0 satisfies

$$(Ar, s^0) \leq 0 \quad \text{all } r \text{ in } R ,$$
$$(h, s^0) \geq 0 \quad \text{all } h \text{ in } P_S^* .$$

Thus s^0 represents a separating hyperplane. Conversely, consider any separating hyperplane H. Since the two convex bodies are themselves in contact (lemma 1), H is a plane of

support to both. A plane of support to a cone necessarily contains the vertex of the cone: in the case of P_S^* this point is the origin. Hence we may represent H as a homogeneous linear functional $s = (s_0, \ldots, s_n)$, some $s_j \neq 0$, such that $(h, s) = 0$ for each point h of H. With proper choice of sign we may then write:

$$(1.5) \qquad (Ar, s) \leq 0 \qquad \text{all } Ar \text{ in } AR,$$

$$(1.6) \qquad (h, s) \geq 0 \qquad \text{all } h \text{ in } P_S^* .$$

By (1.6), s is a point of $(P_S^*)^* = P_S$. Since $s \neq 0$, normalization by a positive factor will yield a unique point in the bounded cross section S of the cone P_S. By (1.5) s is a point of P_{S0}, so that the normalized point is an optimal strategy of player II. Clearly, distinct planes lead to distinct strategies, and vice versa, making the correspondence biunique.

The two lemmas (and their obvious counterparts in terms of SA and P_R^*) give a geometric significance to the sets of optimal strategies, and will enable us to establish a variety of dimensional relationships between them. For sharp results we shall require detailed information about the boundaries of the several convex bodies involved.

2. Geometric model of the discrete game.

If R is taken as the simplex $r_i \geq 0$, $i = 0, 1, \ldots, m$, in the m-dimensional hyperplane

$$\sum_{i=0}^{m} r_i = 1$$

of m+1-space, then the cone P_R is the positive orthant $r_i \geq 0$ of m+1-space. The cone is self-reciprocal: $P_R^* = P_R$. If S, n,

etc., are taken similarly, then we obtain a geometric model of the general zero-valued game with finite sets of strategies. Because the regions here are polyhedral an exact accounting of the dimensions of the optimal strategy sets can be given [1], [4].

3. Polynomial-like games.

We shall be concerned henceforth with games defined by a kernel of the following form:

$$K(x,y) = \sum_{i,j=1}^{m,n} a_{ij}\, r_i(x) s_j(y) \tag{3.1}$$

where the functions r_i and s_j are continuous, and x and y, the strategies of players I and II respectively, are to be chosen from the intervals $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Obviously, any payoff of the form

$$K(x, y) = \sum_{\nu=1}^{N} A_\nu(x) B_\nu(y)$$

can be put in the form (3.1), and conversely. The form (3.1) permits the greatest variety in a geometric model with dimensionality restricted by fixed m and n. The terminology "polynomial-like", "moments", is intended to suggest that the r_i and s_j be regarded as sets of orthogonal polynomials, or trigonometric functions, or simply as the powers x^i, y^j. (The last-named specialization is of course the ultimate object of the present paper.) However no such limitation on the natures of $r_i(x)$ and $s_j(y)$ will actually be demanded.

Mixed strategies may be represented by cumulative probability distribution functions $f(x)$ and $g(y)$, for players I and II respectively, characterized by being monotonic increasing and continuous to the right, with $f(-0) = g(-0) = 0$, $f(1) = g(1) = 1$. The mixed strategy payoff is then written as the Stieltjes integral:

$$(3.2)\quad \int_0^1 \int_0^1 K(x,y)\,df(x)\,dg(y) = \sum_{i,j=1}^{m,n} a_{ij} \int_0^1 r_i(x)\,df(x) \int_0^1 s_j(y)\,dg(y) \quad .$$

Clearly, the only significant properties of the distribution functions are their "moments":

$$(3.3)\qquad r_i = \int_0^1 r_i(x)\,df(x) \qquad s_j = \int_0^1 s_j(y)\,dg(y)$$

for $i = 1,\dots m$, $j = 1,\dots n$. The substitution of (3.3) into (3.2) reveals that the polynomial game is equivalent to the bilinear game (1.1) of § 1 , provided that the sets R and S are convex.

We now characterize the "moment spaces" R and S . The theorem is stated only for R , the set S being entirely analogous. We shall need the following lemma, established by Fenchel [3] .

LEMMA 3. If D is the convex closure of an arbitrary set C in n-space, then every point of D may be represented as a convex combination of at most n+1 points of C. If C is connected, then not more than n points are necessary.

THEOREM 2. R is the convex set spanned by the curve C traced out in m dimensions by

$$r_i = r_i(x)$$

as x varies between 0 and 1.

Proof: Let D be the convex set spanned by C , and suppose a point r^0 of R , is not in D . Then there exists some hyperplane h strictly separating r^0 from D . That is, for some fixed $\delta > 0$,

$$(3.4) \qquad \sum_{i=1}^{m} h_i r_i^0 - \sum_{i=1}^{m} h_i \, r_i(x) \geq$$

for any x in $0 \leq x \leq 1$. Since r^0 is in R there is a distribution function $f^0(x)$ having the "moments" r_i^0. Integrating both sides of (3.4) with respect to $f^0(x)$ we obtain

$$(3.5) \qquad \sum_{i=1}^{m} h_i r_i^0 \int_0^1 df^0(x) - \sum_{i=1}^{m} h_i \int_0^1 r_i(x) df^0(x) \geq \int_0^1 df^0(x) \, ,$$

the inequality holding good because (3.4) is true for all x in the range of the integration. But with the aid of (3.3), (3.5) reduces to

$$\sum_{i=1}^{m} h_i r_i^0 - \sum_{i=1}^{m} h_i r_i^0 \geq \delta \, ,$$

giving the contradiction $0 > 0$. This proves that all points of R are in D.

Conversely suppose r^0 to be in D, and thereby to have a convex representation

$$r_i^0 = \sum_{k=1}^{N} \alpha_k^0 r_i(x_k) \qquad i = 1, 2, \ldots, m,$$

where the α_k^0 are positive and add up to 1, with $N \leq m$ by virtue of lemma 3. It is easily seen that the distribution

$$f^0(x) = \sum_{k=1}^{N} \alpha_k^0 I(x-x_k)$$

has the moments r_i^0, where $I(x-x_k)$ denotes the distribution function putting full weight on $x = x_k$. Hence every point of

D is in R , and the theorem is established.

THEOREM 3. <u>In the polynomial-like game described, both players have optimal mixed strategies with at most</u> min (m, n) steps.

Proof (on player I): In the preceding proof it was observed that every point of R corresponds to a step-function distribution with $N \leq m$ steps. On the other hand, since AR is convexly spanned in n dimensions by the connected set AC , each point of AR is the image under A of a point of R spanned by at most n points of C . By lemma 1 any r in R whose image is a point of $AR \cap P_3^*$ is optimal, hence some mixed strategy having not more than min (m, n) steps is optimal.

COROLLARY. <u>If, in place of (3.1)</u> ,

$$K(x,y) = \sum_{j=1}^{\infty} \sum_{i=1}^{m} a_{ij}\, r_i(x) s_j(y),$$

<u>the convergence being uniform in</u> y , <u>then both players possess optimal mixed strategies with at most</u> m <u>jumps</u>.

Proof: As a result of the uniform convergence, the functions

$$s_i'(y) = \sum_{j=1}^{\infty} a_{ij} s_j(y) \qquad i = 1, 2, \ldots m .$$

are continuous. The kernel may therefore be rewritten:

$$K(x,y) = \sum_{i=1}^{m} r_i(x) s_i'(y) = \sum_{i,j=1}^{m} \delta_{ij} r_i(x) s_j'(y) ,$$

(δ_{ij}) being the unit matrix, and the conclusion of the corollary follows from the theorem.

Letting both sums become infinite destroys in general the discrete nature of the optimal mixed strategies. L. J. Savage has pointed out a class of analytic kernels for which the unique optimal mixed strategies are absolutely continuous distribution functions.

THEOREM 4. *If the dimensions (in R and S) of the sets of optimal mixed strategies are μ and ν respectively, and if ρ is the rank of the matrix (a_{ij}), $(i = 1, \ldots, m,\ j = 1, \ldots, n)$; then there exists an optimal mixed strategy for player I with at most*

$$\min(\rho,\ n - \nu + 1)$$

steps, and for player II with at most

$$\min(\rho,\ m - \mu + 1)$$

steps.

Proof (on player I): As in the proof of the preceding theorem we analyze the convex representation in AR of the points of contact with the cone P_S^*. The convex set AR is ρ-dimensional; hence by lemma 3 every point is spanned by at most ρ points of the connected set AC. Furthermore every point of the contact set $AR \cap P_S^*$ lies in the $n-\nu$-dimensional intersection L of the hyperplanes separating the two bodies. The contact set must be contained in the convex closure of $AC \cap L$. Applying the weaker form of lemma 3, since $AC \cap L$ may not be connected, we obtain a convex representation of any contact point by at most $\min(\rho,\ n - \nu + 1)$ points of AC. The rest of the proof is now evident. This theorem includes Theorem 3, since $\rho \leq \min(m,n)$.

THEOREM 5. *In general*

$$\mu + \nu \leq m + n - \rho,$$

where μ, ν, and ρ are defined as in Theorem 4.

Proof: The set R^0 of optimal points in R has dimension μ. The dimension of AR^0 is at least $\mu - (m - \rho)$, the original dimension less the maximum possible loss due to the degeneracy

of the transformation A. (That is, A is capable of collapsing an $m-\rho$-dimensional set into a point, but nothing more.) On the other hand, AR^0 is the contact set $AR \cap P_S^*$, and lies in $\nu + 1$ linearly independent hyperplanes in $n+1$-space, whose intersection has dimension $n - \nu$. Hence

$$\mu - (m - \rho) \leq \dim (AR^0) \leq n - \nu .$$

COROLLARY. <u>If the matrix</u> (a_{ij}), $i = 1, \ldots, m$, $j = 1, \ldots, n$, <u>is not degenerate, then</u>

$$\mu + \nu \leq \max (m,n).$$

<u>4. Polynomial games: Description of the moment spaces.</u>

The polynomial game with kernel

$$K(x,y) = \sum_{i,j=0}^{m,n} a_{ij} x^i y^j \tag{4.1}$$

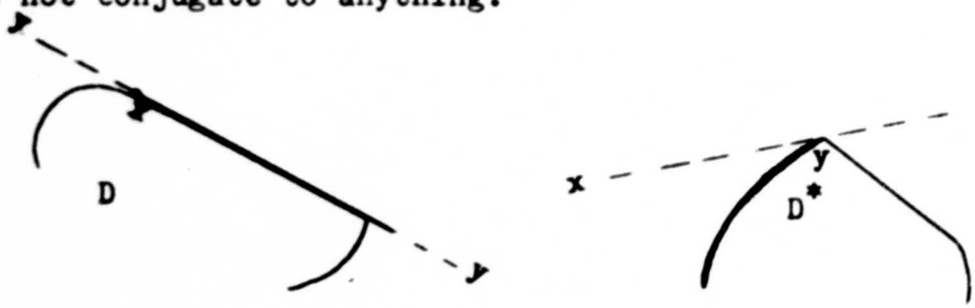
played on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ is a specialization of (3.1) of particular interest. Since $r_0 = s_0 = 1$ identically for all distribution functions, R and S are m- and n-dimensional regions already naturally embedded in $m+1$- and $n+1$-space. After some preparatory discussion we shall give a description in detail of these regions and of their conjugate cones P_R^* and P_S^*, paying special attention to the boundaries as they affect the possible contact sets.

The appearance of coefficients a_{i0} and a_{0j} not zero has not significantl. altered the model. The statement of §2 that AR has dimension at most n is no longer valid: the full number $n + 1$ of dimensions is now possible provided that $n < m$. As before we assume an adjustment of a_{00} to make the value of the game zero.

We define two indices of surface dimension, $a(x)$ and $c(x)$, for a point x in the boundary of a convex set D in n-space, in order to describe the nature of the hypersurface at that point. Let $L(x)$ denote the intersection of all the hyperplanes of support to D that contain x. Let $a(x)$ denote the dimension of $L(x)$, and let $c(x)$ denote the dimension of the convex set in which $L(x)$ meets D. The sets of points for which $a(x) = a$, $0 \leq a < n$, are the a-dimensional components, or "faces", of the boundary of D. The c-index tells in how many directions the boundary is actually flat. (Thus if D is polyhedral then $a(x) = c(x)$ everywhere.) Both indices are affine-invariant, and are not changed by increasing the dimension of the space in which the convex set is imbedded. This last fact suggests the definition $a(x) = c(x) = n$ for points x interior to D.

We shall have occasion to use $a(x)$ and $c(x)$ referring to points x in the boundary of a convex cone. In such context we calculate with respect to a bounded cross section through the cone at x, rather than with respect to the cone itself. The values of course are independent of the choice of cross section. For the vertex of the cone, $a(x) = c(x) = -1$.

If D^* is a bounded cross section of the cone P_D^*, then the points y in the boundary of D^* may be regarded as the supporting planes to D, and vice versa. If $y \in D^*$ is an inner point of all y that are supporting planes to D at a point x, then we say y is <u>conjugate</u> to x. The relationship is not always symmetric: x will be a supporting plane to D^* at y, but not necessarily an inner supporting plane. In the figure, y is conjugate to x, but x is not conjugate to anything.



LEMMA 4. Let x be in the boundary of a convex set D in n-space, and let y be in the boundary of D^*, a bounded cross section of P_D^*. If y is conjugate to x, or x conjugate to y, then,

$$a(x) + c(y) = c(x) + a(y) = n - 1 \tag{4.2}$$

Proof: Take any $x \in D$, $y \in D^*$ with $(x,y) = 0$. Let A be the set of planes of support to D^* at y:

$$A = \left\{x' \in D \mid (x',y) = 0\right\} .$$

There are $n - a(y)$ linearly independent x' in A. The dimension of A, considered as a convex subset in the boundary of D, is therefore $n - a(y) - 1$. On the other hand, the set $L(x) \cap D = B$ has dimension $c(x)$ by definition; while every x' in B satisfies $(x', y) = 0$. Therefore A contains B. Now the hypothesis tells us that either

(a) y is an inner plane of support to D at x, or
(b) x is an inner point of A.

In case (a), y supports D in precisely the set B; that is, (x',y) is actually positive for any x' in $D - B$. Hence B contains A. In case (b), every supporting plane through x must contain all of A (otherwise A would have points on both sides of the plane). Hence $L(x)$ contains A and B contains A. It follows that $B = A$ and $\dim B = \dim A$. This gives the righthand equality of (4.2). The other follows by symmetry, since D is a bounded cross section of $(P_{D^*})^*$.

The following description of the finite dimensional moment spaces is drawn from [6]. The set R is the convex closure of the curve C traced out by

$$r_i = x^i \qquad i = 1, 2, \ldots m$$

as x varies between zero and one (Theorem 1). If $m \geq 2$ then all the points of C are actually extreme (not expressible as convex combinations of other points of R). These points correspond to the pure strategies $I(x - x')$.

Points in the boundary of R have unique representations as convex linear combinations of extreme points. To see this fact, consider the characteristic property of any supporting hyperplane to R:

$$(4.3) \qquad h_0 + \sum_{i=1}^{m} h_i r_i \geq 0 \qquad \text{all } r \text{ in } R.$$

This is equivalent to

$$(4.4) \qquad h_0 + \sum_{i=1}^{m} h_i x^i \geq 0 \qquad \text{all } x \text{ in } 0 \leq x \leq 1.$$

The equality holds for at least one value of x, but not for more than m values, as not all the coefficients of the polynomial (4.4) vanish at once. Thus, the curve C does not touch the hyperplane more than m times. The $k \leq m$ contacts are linearly independent since their coordinates form the Vandermonde matrix

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_k \\ x_1^2 & x_2^2 & & x_k \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_1^m & x_2^m & \cdots & x_k^m \end{pmatrix}$$

whose rank is k. Every supporting hyperplane therefore meets the boundary of R in a simplex, the points of which can be represented in precisely one way as convex combinations of the vertices. But every point in the boundary of R is touched by some supporting hyperplane. Q.E.D.

The smallest number of points of C by which a point r can be spanned we shall denote by $b(r)$. We let $b'(r)$ denote the same number, but with the end points $(0,\dots,0)$ and $(1,\dots,1)$ counted conventionally as half points. Thus $b'(r)$ can take on half-integral values, and always

$$(4.5) \qquad b(r) - 1 \leq b'(r) \leq b(r) .$$

The previous discussion of the contact simplex has effectively established that

$$(4.6) \qquad b(r) = c(r) + 1$$

for r on the boundary of R. If we observe that any root in $0 < x < 1$ of the polynomial (4.4) must be double in order to preserve the inequality, it is then easy to show that

$$(4.7) \qquad 2b'(r) = a(r) + 1.$$

For r interior to R we have the following improvement on lemma 3:

$$(4.8) \qquad 2b'(r) = m + 1;$$

which suggests taking $a(r) = m$ in the interior. (This, together with $c(r) = m$, is the value obtained from the definitions if R is imbedded in a space of higher dimension. Formula (4.6) is not valid under this extension.) On the boundary of R we have, by (4.5), (4.6), (4.7):

$$a(r) - 1 \leq 2c(r) \leq a(r) + 1 .$$

That is, c is approximately half of a.

If the notion of representation by convex linear combinations is extended to permit the use of infinite sets of extreme points, then the convex representations of any r^0 in R correspond one-one to the distributions $f^0(x)$ whose moments are the coordinates of r^0.

The cone P_R is naturally obtained by considering the moment

$$f_0 = \int_0^1 df(x)$$

as an $m+1^{st}$ coordinate r_0 and allowing distributions with arbitrary non-negative total weight $f_0 \geq 0$. The probability distributions of R have of course all had $f_0 = 1$.

The conjugate cone P_R^*, because of the close connection between the linear form (4.3) and the polynomial (4.4), can be regarded as the set of all polynomials, of degree m or less, which are not negative in the interval $0 \leq x \leq 1$. The boundary of P_R^* comprises just those with at least one root in the interval. The extreme points (of a bounded cross section of the cone) are just those with all m roots in the interval. The latter fall into two components according to whether the root 1 occurs with even or odd multiplicity; this is reflected in the sign of the leading coefficient h_m. Every point of the cross section lies on some line segment connecting the two components of extreme points; that is, every point is spanned by two extreme points. A convenient bounded cross section, which we call R^*, is obtained by the normalization:

$$\sum_{i=0}^{m} h_i/(i+1) = \int_0^1 \sum_{i=0}^{m} h_i x^i \, dx = 1 .$$

Much of the foregoing material has been given for its conceptual interest, without proof, since it makes no contribution to the rigor of the derivations of the next section. Detailed proofs, and many ramifications and applications of this material, will appear in [6].

5. Polynomial games; The sets of optimal strategies.

We now proceed to derive the main results. Let μ and ν denote, as before, the dimensions of the sets of optimal moment strategies for players I and II respectively. Let ρ be the rank of the $m \times n+1$ matrix (a_{ij}) omitting $i = 0$; σ the rank of the $m+1 \times n$ matrix (a_{ij}) omitting $j = 0$. (Note that the change in a_{00} to make the game value zero does not affect these ranks.) The dimension of AR is then precisely ρ. Our attention will be chiefly directed to the "general" case $\rho = \min(m, n+1)$, $\sigma = \min(m+1, n)$.

In particular, let $\rho = m$, so that the transformation A is non-singular. All the dimensional indices are preserved under A: $a(r) = a(Ar)$, $c(r) = c(Ar)$. Take an optimal point r^0 such that Ar^0 is an inner point of the contact set $AR \cap P_S^*$. Take s^0 to be an inner point of the set $S^0 \subsetneq S$ of optimal strategies of player II. Ar^0 and s^0 are "almost always" conjugate. However it can happen that neither is conjugate to the other. To cover such cases we shall need a point on the surface of the cone that is conjugate to s^0; we shall call this point s^*. Such a point always exists if s^0 is in the boundary of S; if s^0 is interior we define the vertex 0 of the cone to be conjugate to s^0, observing that lemma 4 is valid with the convention $a(s^0) = c(s^0) = n$, $a(0) = c(0) = -1$.

Several dimensional inequalities may be derived:

(a) Directly from the definitions we have

$$\mu \leq c(r^0), \qquad \nu \leq c(s^0) . \tag{5.1}$$

(b) Every plane of support to P_S^* at s^* contains the contact set $AR \cap P_S^*$. But the dimension of the contact set is just μ. Hence, allowing for contact along the conical elements of P_S^*, we have $c(s^*) + 1 \geq \mu$, or, by lemma 4,

$$\mu \leq n - a(s^0) . \tag{5.2}$$

(c) There are $n+1 - a(r^0)$ independent supporting hyperplanes to AR at the point Ar^0. The ν- dimensional family of separating planes all support AR at that point; hence,

$$(5.3) \qquad \nu \leq n - a(r^0),$$

since the number of linearly independent items is one more than the dimension number.

(d) On the other hand, there are <u>at least</u> $n - a(s^*)$ independent hyperplanes through the origin supporting the cone P_S^* at the point Ar^0, since, as remarked in (b) above, every supporting plane at s^* contains the contact set. To weed out the hyperplanes that do not separate, we must only require that they support AR as well. This imposes <u>at most</u> $a(r^0) - \mu$ further constraints, leaving at least $n - a(s^*) - a(r^0) + \mu$ independent separating planes. Hence,

$$\nu \geq n - a(s^*) - a(r^0) + \mu - 1;$$

or, using lemma 4,

$$(5.4) \qquad \mu - \nu \leq a(r^0) - c(s^0).$$

(e) The intersection of the separating planes has dimension $n - \nu$. Each separating plane contains s^* and Ar^0, in the boundaries of P_S^* and AR respectively; and hence contains sets with dimension $c(s^*)+1$ and $c(r^0)$. The common part of these sets is at most μ-dimensional. Therefore

$$n - \nu \geq c(s^*)+1 + c(r^0) - \mu.$$

Using lemma 4 this becomes:

$$(5.5) \qquad \mu - \nu \geq c(r^0) - a(s^0).$$

Despite the symmetry of (5.4) and (5.5), separate derivations were necessitated by our one-sided hypothesis $\rho = m$.

The inequalities (5.1) through (5.5) may be converted with the help of (4.6) - (4.8) into statements about the more palpable (from the standpoint of the game) quantities $b(r^0)$, $b'(r^0)$, etc. For we may define a pure strategy x as <u>essential</u> if there is an optimal mixed strategy in which x is played.[4] Then if r^0 is in the boundary of R player I has precisely $b(r^0)$ essential pure strategies, by the uniqueness of the convex representations of boundary points. On the other hand, if r^0 is interior to R then every pure strategy is essential. We sum up:

THEOREM 6. *In the polynomial game with kernel*

$$K(x,y) = \sum_{i,j=0}^{m,n} a_{ij} x^i y^j \qquad m \leq n+1$$

in which the $m \times n+1$ *matrix* (a_{ij}), $i \neq 0$, *has rank* m, *the mixed strategies are representable as points in the* m- *and* n-*dimensional moment spaces* R *and* S. *If* r^0 *is an inner point of the* μ-*dimensional convex set of optimal moment strategies for player I; if similarly* s^0 *and* ν *for player II, then*

$$\begin{aligned} \mu &\leq n + 1 - 2b'(s^0) \\ \nu &\leq n + 1 - 2b'(r^0) \\ \mu - \nu &\leq 2b'(r^0) - b(s^0) \\ \nu - \mu &\leq 2b'(s^0) - b(r^0) \ . \end{aligned}$$

Also, if r^0 *is in the boundary of* R, *then*

$$\mu \leq b(r^0) - 1$$

and the number of essential pure strategies for I is exactly $b(r^0)$.
Similarly, if s^0 *is in the boundary of* S, *then*

$$\nu \leq b(s^0) - 1$$

<u>and the number of essential pure strategies for II is</u> $b(s^0)$.

The approximate equality (4.5) of b and b' should be recalled in interpreting this theorem.

<u>Interior optimal moment strategies</u>. The case in which player II has an optimal s^0 interior to S merits special notice. Here the origin of $n+1$-space is the sole contact point of the two convex bodies. From (5.2) we now get the precise result $\mu = 0$, since $a(s^0) = c(s^0) = n$. Moreover the argument of (d) now counts exactly the number of constraints on the family of separating planes, and yields

$$\nu = n - a(r^0)$$

(which may also be derived from a sharpening of the argument of (c)). We have established:

THEOREM 7. <u>In the polynomial game of Theorem 5, if</u> s^0 <u>is interior to</u> S, <u>then player I's optimal moment strategy</u> r^0 <u>is unique</u>, and

$$\nu = n + 1 - 2b'(r^0) \ .$$

Still retaining the one-sided condition $\rho = m$, we now suppose that player I has an interior optimal moment strategy. In this case m must be less than $n+1$ in order to expose the inner points of AR to contact. But substituting $a(r^0) = c(r^0) = m$ into (5.1) – (5.5) does not lead beyond the inequalities of Theorem 6. In other words, the assumption of an interior optimal strategy, for the player with the greater number of dimensions to play with, is quite strong (Theorem 7); while the same assumption for the other player is hardly restrictive at all.

In the square case, $m = n = \rho$, if s^0 is interior and unique, then r^0 is also interior and unique, and

$$b'(r^0) = b'(s0) = (n + 1)/2 \ .$$

There is a simple construction for games of this type, which is merely a matter of finding a nonsingular, $n+1 \times n+1$ matrix A such that

$$Ar^0 = s^0 A = (v, 0, \ldots, 0),$$

where r^0 and s^0 are the desired interior moment strategies and $v \neq 0$ is the desired value.

Equalizing optimal strategies. If it happens that one player has an optimal strategy which makes the outcome independent of his opponent's action, then a notable simplification occurs in the computation of the solution (see the next section). For example, the pure strategies x^0 and y^0 are equalizing optimal strategies if the payoff can be written in the form

$$K(x,y) = P(x)Q(y)R(x\ ,y) + k,$$

with x^0 a root of P, y^0 a root of Q. The geometrical significance, if player I has an optimal equalizer, is that the origin lies in AR. For player II it means that some plane of support to P_S^* contains the whole set AR. Existence of an interior solution for one player requires that all optimal strategies of his opponent be equalizers. For an interior moment strategy may be realized as a convex combination of pure strategies in which given pure strategy appears with positive weight. Use of this pure strategy against any opposing optimal strategy can only give the value of the game as payoff. Hence every opposing optimal strategy must be an equalizer. (Compare [1] , lemma 2.) The converse is not true; all of one player's optimal strategies may be equalizers without the other player having an interior moment strategy which is optimal. This contrasts with a property of finite games: that only the essential pure strategies yield the value of the game against every opposing optimal strategy. ([1], Theorem 1.)

The foregoing considerations are valid as well for polynomial-like games, with "inner solution" for "interior solution", since the full dimensionality of the moment space does not figure in the argument. No rank restrictions need be placed on (a_{ij}).

We note in conclusion that an equalizing strategy is not <u>a fortiori</u> an optimal strategy. This fact somewhat reduces the value of the "equalizer" concept, so far as computation is concerned (see below).

6. Polynomial games: Computation.

This brief section is included to indicate the order of magnitude of the computational difficulties, when the solution of a polynomial game is tackled directly. There is no discussion of approximation methods.

It is possible to reduce the solution of the game problem to the solution of certain systems of algebraic equations — linear in some cases, non-linear in the remaining.

(a) If there exists an equalizing $g^0(y)$ such that

$$(6.1) \qquad \sum_{i,j=0}^{m,n} \int_0^1 a_{ij} x^i y^j dg^0(y) = w \qquad \text{for all } x ,$$

then, since the moment $g_0^0 = 1$, we have

$$(6.2) \qquad a_{00} + \sum_{j=1}^{n} a_{0j} g_j^0 = w$$

$$(6.3) \qquad a_{i0} + \sum_{j=1}^{n} a_{ij} g_j^0 = 0 \qquad i = 1, 2, \ldots, m.$$

(The g_j here are of course the s_j of earlier sections. The emphasis is now on the distribution functions rather than on the geometry.) The equations (6.3), solved for g_j^0, cannot give the moments of an equalizer unless the rank ρ of the matrix (a_{ij}), $i \neq 0$,

is n or less. More generally, if $\rho > n - \nu$ then no equalizing g^0 exists, as a simple contradiction shows, and it is useless to proceed with this attack.

A solution g_j^0 of (6.3) is a set of moments if and only if the two quadratic forms

$$\sum_{k,\ell=0}^{n'} g^0_{k+\ell} x_k x_\ell , \qquad \sum_{k,\ell=0}^{n'-1} (g^0_{k+\ell+1} - g^0_{k+\ell+2}) x_k x_\ell \quad (\text{if } n = 2n') ,$$

$$\sum_{k,\ell=0}^{n'} g^0_{k+\ell+1} x_k x_\ell , \qquad \sum_{k,\ell=0}^{n'} (g^0_{k+\ell} - g^0_{k+\ell+1}) x_k x_\ell \quad (\text{if } n = 2n'+1),$$

are non-negative definite (compare [8] p. 77). The equalizer $g^0(y)$ is then a solution of the game only if the constant w, given by (6.2), is in fact the value of the game. This can ordinarily be established only by the discovery of an optimal $f^0(x)$. In particular, if the solution of

$$a_{0j} + \sum_{i=1}^{m} a_{ij} f_i^0 = 0 \qquad j = 1, 2, \ldots, n$$

is a set of moments as well, then w is automatically the value, and the equalizers $f^0(x)$, $g^0(y)$ solve both the given game and its negative.

The problem of reconstructing a distribution function from a given (finite) set of moments is essentially equivalent to that of obtaining the roots of a certain polynomial, closely related to the quadratic forms above, whose degree is approximately half the number of moments given.[6]

(b) In general, the process (a) will not solve the game unless equalizers exist for both players. If player II, say, has no equalizer, then the polynomial (6.1) can reach its maximum in

$0 \leq x \leq 1$ at not more than $m' = [m/2] + 1$ points x_k (or $m/2$ points in the "b'" system of counting). The number of essential strategies for I is limited thereby to m', and his optimal distribution has the form:

$$f^{0}(x) = \sum_{k=1}^{m'} \alpha_k I(x - x_k) .$$

Non-uniqueness may appear in the α_k, but not in the x_k.

The sets of moments will have the form

$$f_i^{0} = \sum_{k=1}^{m'} \alpha_k x_k^{i} , \qquad i = 1, 2, \ldots, m,$$

$$g_j^{0} = \sum_{\ell=1}^{n'} \beta_\ell y_\ell^{j} \qquad j = 1, 2, \ldots, n,$$

where $n' = [n/2] + 1$. The polynomials

$$H^{0}(x) = \sum_{i,j=0}^{m,n} a_{ij} g_j^{0} x^{i} , \qquad K^{0}(y) = \sum_{i,j=0}^{m,n} a_{ij} f_i^{0} y^{j}$$

must satisfy the equations

$$\left.\begin{array}{l} H^{0}(x_k) = w_1 \\ K^{0}(y_\ell) = w_2 \end{array}\right\} \ldots\ldots\ldots\ldots \text{(all } k, \ell\text{)},$$

$$\left.\begin{array}{l} \frac{d}{dx} H^{0}(x_k) = 0 \\ \frac{d}{dy} K^{0}(y_\ell) = 0 \end{array}\right\} \ldots\ldots\ldots\ldots \text{(all } k, \ell \text{ except ends)}.$$

The four possible arrangements of essential strategies at the points 0 and 1 must be tried separately in solving these equations, and the condition on the derivative omitted in those cases where x_k or y_ℓ is fixed equal to 0 or 1 . In each case there turn out to be $m + n + 2$ unknowns α_k, β_ℓ, x_k, y_ℓ , and w_h. Together with the normalizations

$$\sum_{k=1}^{m'} \alpha_k = 1 , \qquad \sum_{\ell=1}^{n'} \beta_\ell = 1 ,$$

there are also $m + n + 2$ equations, which are linear in the α_k, β_ℓ, w_h. Every solution of this system must give $w_1 = w_2$, as a simple argument shows. At least one solution exists in which all of the other unknowns lie between 0 and 1 (thereby giving legitimate distribution functions $f^o(x)$, $g^o(y)$), and having the "saddle point" property:

$$H^o(x) \leq w_1 = w_2 \leq K^o(y) \tag{6.4}$$

for all $0 \leq x \leq 1$, $0 \leq y \leq 1$. All solutions of this type are solutions of the game. In general there will be many other solutions of the equations which do everything but satisfy (6.4). These will locate maxima and minima of the original kernel (4.1), solve the negative of the given game, and perform other more obscure tasks. It is not until (6.4) that we make use of the primary motivations: of player I to maximize the kernel, and of player II to minimize.

In the foregoing treatment we have assumed the worst, i.e., that each player had the greatest possible (finite) number of essential pure strategies. A more practical approach might be to start with small values of m' and n', and work up.

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